International Journal of Applied Mathematics

Volume 25 No. 3 2012, 309-325

HARMONIC MEROMORPHIC FUNCTIONS INVOLVING CERTAIN m-TUPLE INTEGRAL OPERATORS

Poonam Sharma

Department of Mathematics and Astonomy
University of Lucknow
Lucknow, 226007, INDIA
sharma_poonam@lkouniv.ac.in

Abstract: The purpose of this paper is to study certain mappings of a harmonic meromorphic function $\mathbf{F} = \mathbf{H} + \overline{\mathbf{G}}$ which involve m-tuple integral operators defined in the exterior of the unit disk $\widetilde{\mathcal{U}}$. Necessary and sufficient coefficient conditions for $\mathbf{F}$ to be in the classes $\mathcal{MH}^*(\gamma)$ and $\mathcal{KH}(\gamma)$, respectively, are obtained. Further, some *Wright's generalized hypergeometric (Wgh) functions inequalities* which are also necessary and sufficient conditions for $\mathbf{F}$ to be in these classes are derived. Results on bounds and extreme points are also discussed.

AMS Subject Classification: 30C45, 30C50

Key Words: harmonic meromorphic functions, harmonic starlike (convex) functions, distortion bounds, extreme points

1. Introduction

A continuous function $f = u + iv$ is called a complex valued harmonic map in a simply connected domain $\mathcal{D} \subset \mathbb{C}$ if u and v are real valued harmonic functions in $\mathcal{D}$. Clunie and Sheil-Small [3] introduced a class $\mathcal{SH}$ of complex valued harmonic maps f which are univalent and sense-preserving in the open

Received: September 21, 2011

unit disk $\mathcal{U} = \{z : z \in \mathbb{C}, |z| < 1\}$ and assume a normalized representation $f = h + \overline{g}$ where

$$h(z) = \sum_{n=1}^{\infty} h_n z^n,\ h_1 = 1;\ g(z) = \sum_{n=1}^{\infty} g_n z^n,\ |g_1| < 1$$

are analytic and univalent in the open unit disk $\mathcal{U}$. Also it is proved by Clunie and Sheil-Small [3], that the function $f = h + \overline{g} \in \mathcal{SH}$ is locally univalent and sense-preserving in $\mathcal{U}$ if and only if the Jacobian $J_f(z) = \left|h'(z)\right|^2 - \left|g'(z)\right|^2 > 0$ $(z \in \mathcal{U})$. Hengartner and Schober in [5] studied a class of complex valued harmonic functions f which are orientation preserving and univalent in exterior of the unit disk $\widetilde{\mathcal{U}} = \{z : z \in \mathbb{C}, |z| > 1\}$ with $f(\infty) = \infty$. Such functions admit the representation

$$f(z) = h(z) + \overline{g(z)} + A \log |z|, \tag{1.1}$$

where

$$h(z) = \alpha z + \sum_{n=1}^{\infty} a_n\, z^{-n}; \quad g(z) = \beta z + \sum_{n=1}^{\infty} b_n\, z^{-n} \tag{1.2}$$

are analytic in $\widetilde{\mathcal{U}}$, $\alpha, \beta, A \in \mathbb{C}$ with $0 \le |\beta| < |\alpha|$, and $w(z) := \overline{f_{\overline{z}}}/f_z$ is analytic with $|w(z)| < 1$ for $z \in \widetilde{\mathcal{U}}$. Jahangiri and Silverman in [6] (also Jahangiri in [7]) removed the logarithmic singularity by letting $A = 0$ in (1.1) and considered a class $\mathcal{MH}$ of functions:

$$f(z) = h(z) + \overline{g(z)}, \tag{1.3}$$

where $h(z)$ and $g(z)$ are of the form (1.2).

Denote by $\overline{\mathcal{MH}}$, a subclass of $\mathcal{MH}$ consisting of functions f of the form (1.3) for which the functions h and g are restricted by

$$h(z) = |\alpha|\, z + \sum_{n=1}^{\infty} |a_n|\, z^{-n}; \quad g(z) = |\beta|\, z - \sum_{n=1}^{\infty} |b_n|\, z^{-n}. \tag{1.4}$$

A necessary and sufficient condition for $f \in \mathcal{MH}$ to be starlike of order $\gamma, 0 \le \gamma < 1$ in $\widetilde{\mathcal{U}}$ is that

$$\frac{\partial}{\partial\theta} \arg f(re^{i\theta}) \ge \gamma, z = re^{i\theta}, 0 \le \theta < 2\pi, r > 1.$$

A necessary and sufficient condition for $f \in \mathcal{MH}$ to be convex of order $\gamma, 0 \le \gamma < 1$ in $\widetilde{\mathcal{U}}$ is that

$$\frac{\partial}{\partial\theta} \arg \left(\frac{\partial}{\partial\theta} f(re^{i\theta})\right) \ge \gamma, z = re^{i\theta}, 0 \le \theta < 2\pi, r > 1.$$

The classes of such starlike and convex functions of order γ are, respectively, denoted by $\mathcal{MH}^*(\gamma)$ and $\mathcal{KH}(\gamma)$. Furthermore, $\overline{\mathcal{MH}}^*(\gamma) \equiv \mathcal{MH}^*(\gamma) \cap \overline{\mathcal{MH}}$ and $\overline{\mathcal{KH}}(\gamma) \equiv \mathcal{KH}(\gamma) \cap \overline{\mathcal{MH}}$.

2. Preliminaries and Definitions

From the work of Jahangiri [7], we state following results.

Lemma 1. *Let $f = h + \overline{g}$, h and g be of the form (1.2). If*

$$\sum_{n=1}^{\infty} \left((n+\gamma)|a_n| + (n-\gamma)|b_n|\right) \leq (1-\gamma)|\alpha| - (1+\gamma)|\beta|, \tag{2.1}$$

for $0 \leq \gamma < \frac{|\alpha|-|\beta|}{|\alpha|+|\beta|}$, then f is harmonic, univalent, orientation-preserving in $\widetilde{\mathcal{U}}$ and $f \in \mathcal{MH}^*(\gamma)$. The condition (2.1) is necessary if $f \in \overline{\mathcal{MH}}^*(\gamma)$.

Lemma 2. *Let $f = h + \overline{g}$, h and g be of the form (1.2). If*

$$\sum_{n=1}^{\infty} n\left((n+\gamma)|a_n| + (n-\gamma)|b_n|\right) \leq (1-\gamma)|\alpha| - (1+\gamma)|\beta|, \tag{2.2}$$

for $0 \leq \gamma < \frac{|\alpha|-|\beta|}{|\alpha|+|\beta|}$, then $f \in \mathcal{KH}(\gamma)$. The condition (2.2) is necessary if $f \in \overline{\mathcal{KH}}(\gamma)$.

The case for $\gamma = 0, \alpha = 1, \beta = 0$ is studied in [6].

Inspired by the work of Kiryakova [9], [10], in this paper, we define an integral operator $J_{\beta_1}^{\nu_1,\delta_1} h(z)$ for functions $h(z), z \in \widetilde{\mathcal{U}}$ of the form (1.2) and for any real ν_1 and positive β_1 such that $\nu_1 + 1 > \beta_1$, as follows:

$$\begin{aligned} J_{\beta_1}^{\nu_1,\delta_1} h(z) &= \frac{z^{\frac{\nu_1+1}{\beta_1}}}{\Gamma(\delta_1)} \int_z^{\infty} \left(\sigma^{\frac{1}{\beta_1}} - z^{\frac{1}{\beta_1}}\right)^{\delta_1 - 1} \sigma^{-\frac{(1+\delta_1+\nu_1)}{\beta_1}} h(\sigma)\, d(\sigma^{\frac{1}{\beta_1}}), \quad \delta_1 > 0 \\ &= \frac{1}{\Gamma(\delta_1)} \int_0^1 (1-t)^{\delta_1 - 1} t^{\nu_1} h(z t^{-\beta_1}) dt, \\ J_{\beta_1}^{\nu_1,0} h(z) &= h(z)\ . \end{aligned} \tag{2.3}$$

The integral defined in (2.3) for $\delta_1 > 0$ is convergent and the image of power function $z^k, z \in \widetilde{\mathcal{U}}$ under this operator with the use of the beta function is given by

$$J_{\beta_1}^{\nu_1,\delta_1} z^k = \frac{\Gamma(\nu_1 + 1 - k\beta_1)}{\Gamma(\nu_1 + \delta_1 + 1 - k\beta_1)} z^k,$$

for some $k < \frac{\nu_1+1}{\beta_1}$.

Further, similarly to the multiple (m-tuple) integral operator introduced and studied by Kiryakova in [9] (see also equ. (12), p. 13, [10]), we define for some $m \in \mathbb{N} = \{1, 2, 3, ...\}$, the m-tuple integral operators $J_{(\beta_i),m}^{(\nu_i),(\delta_i)} h(z)$ and $J_{(\beta_i^{'}),m}^{(\nu_i^{'}),(\delta_i^{'})} g(z)$ by repeating m-times the integral operators of type (2.3), for $h(z)$ and $g(z)$ of the form (1.2) and for $\delta_i, \delta_i^{'} \in \mathbb{R}_+ \cup \{0\}$, $\beta_i, \beta_i^{'} \in \mathbb{R}_+, \nu_i, \nu_i^{'} \in \mathbb{R}$, such that $\nu_i + 1 > \beta_i, \nu_i^{'} + 1 > \beta_i^{'}$, $\forall\, i = 1, 2, ..., m$, as follows:

$$J_{(\beta_i),1}^{(\nu_i),(\delta_i)} h(z) = J_{\beta_1,1}^{\nu_1,\delta_1} h(z) = J_{\beta_1}^{\nu_1,\delta_1} h(z), \delta_1 > 0; J_{\beta_1}^{\nu_1,0} h(z) = h(z), \quad z \in \widetilde{\mathcal{U}},$$

$$J_{(\beta_i),2}^{(\nu_i),(\delta_i)} h(z) = J_{\beta_2}^{\nu_2,\delta_2}\ J_{\beta_1}^{\nu_1,\delta_1} h(z), \delta_1 + \delta_2 > 0; J_{(\beta_i),2}^{(\nu_i),(0)} h(z) = h(z), \quad z \in \widetilde{\mathcal{U}}$$

and hence, for any $i = 1, 2, ..., m, m \in \mathbb{N}$,

$$J_{(\beta_i),m}^{(\nu_i),(\delta_i)} h(z) = \left[\prod_{i=1}^{m} J_{\beta_i}^{\nu_i,\delta_i}\right] h(z), \sum_{i=1}^{m} \delta_i > 0;\ J_{(\beta_i),m}^{(\nu_i),(0)} h(z) = h(z),$$
$$z \in \widetilde{\mathcal{U}}, \quad (2.4)$$

$$J_{(\beta_i^{'}),m}^{(\nu_i^{'}),(\delta_i^{'})} g(z) = \left[\prod_{i=1}^{m} J_{\beta_i^{'}}^{\nu_i^{'},\delta_i^{'}}\right] g(z), \sum_{i=1}^{m} \delta_i > 0; J_{(\beta_i^{'}),m}^{(\nu_i^{'}),(0)} g(z) = g(z), \quad z \in \widetilde{\mathcal{U}},$$

the symbol $(\lambda_i) = (\lambda_1, \lambda_2, ..., \lambda_m)$ for any $i = 1, 2, ..., m, m \in \mathbb{N}$ with $(\lambda_1) = \lambda_1$ is called m-tuple and (0) denotes the m-tuple with all its m components are equal to 0 (for details about the symbols used one can refer to Kiryakova [10]).

Note that the integral operator $J_{\beta_1}^{\nu_1,\delta_1} h(z)$ is an analogue of the Erdélyi-Kober integral operator [9] (see also eq. (1), p. 11, [10]) which is defined for the functions analytic in the unit disk $\mathcal{U}$ and is studied as the multiple (m-tuple) Erdélyi-Kober integral operators by Kiryakova in [10]-[12].

Involving m-tuple integral operators defined in (2.4) for $h(z)$ and $g(z)$ of the form (1.2), we consider for some $m \in \mathbb{N}$, $\delta_i, \delta_i^{'} \in \mathbb{R}_+ \cup \{0\}$, $\beta_i, \beta_i^{'} \in \mathbb{R}_+, \nu_i, \nu_i^{'} \in \mathbb{R}$, such that $\nu_i + 1 > \beta_i, \nu_i^{'} + 1 > \beta_i^{'}$, $\forall\, i = 1, 2, ..., m$, an harmonic function:

$$\mathbf{F}(z) = \mathbf{H}(z) + \overline{\mathbf{G}(z)} \in \mathcal{MH}, \tag{2.5}$$

where

$$\begin{aligned} \mathbf{H}(z) &= J_{(\beta_i),m}^{(\nu_i),(\delta_i)} h(z) = \alpha \lambda_1 z + \sum_{n=1}^{\infty} \theta_n \; a_n \; z^{-n}, \qquad (2.6) \\ \mathbf{G}(z) &= J_{(\beta_i^{'}),m}^{(\nu_i^{'}),(\delta_i^{'})} g(z) = \beta \lambda_1^{'} z + \sum_{n=1}^{\infty} \theta_n^{'} \; b_n \; z^{-n}, \lambda_1^{'} < \lambda_1, \end{aligned}$$

$$\begin{aligned} \theta_n &= \prod_{i=1}^{m} \frac{\Gamma(\nu_i + 1 + n\beta_i)}{\Gamma(\nu_i + \delta_i + 1 + n\beta_i)}, \theta_n^{'} = \prod_{i=1}^{m} \frac{\Gamma\left(\nu_i^{'} + 1 + n\beta_i^{'}\right)}{\Gamma\left(\nu_i^{'} + \delta_i^{'} + 1 + n\beta_i^{'}\right)}, n \geq 1, \qquad (2.7) \\ \lambda_1 &= \prod_{i=1}^{m} \frac{\Gamma(\nu_i + 1 - \beta_i)}{\Gamma(\nu_i + \delta_i + 1 - \beta_i)}, \lambda_1^{'} = \prod_{i=1}^{m} \frac{\Gamma\left(\nu_i^{'} + 1 - \beta_i^{'}\right)}{\Gamma\left(\nu_i^{'} + \delta_i^{'} + 1 - \beta_i^{'}\right)}. \end{aligned}$$

In order to get some Wgh inequalities, we recall here the Wright's generalized hypergeometric (Wgh) function defined in [13] for

$$a_i \in \mathbb{C} \left(\frac{a_i}{A_i} \neq 0, -1, -2, ...; i = 1, 2, ..., p \right)$$

and

$$b_i \in \mathbb{C} \left(\frac{b_i}{B_i} \neq 0, -1, -2, ...; i = 1, 2, ..., q \right)$$

with $A_i > 0 \; (i = 1, 2, ..., p), B_i > 0 \; (i = 1, 2, ..., q)$ satisfying $1 + \sum_{i=1}^{q} B_i - \sum_{i=1}^{p} A_i \geq 0$ as follows:

$$\begin{aligned} & {}_p\Psi_q \left[\begin{array}{c} (a_1, A_1), ..., (a_p, A_p) \\ (b_1, B_1), ..., (b_q, B_q) \end{array} ; z \right] \\ & \qquad = {}_p\Psi_q \left[\begin{array}{c} (a_i, A_i)_{1,p} \\ (b_i, B_i)_{1,q} \end{array} ; z \right] = \sum_{n=0}^{\infty} \frac{\prod_{i=1}^{p} \Gamma(a_i + nA_i)}{\prod_{i=1}^{q} \Gamma(b_i + nB_i)} \frac{z^n}{n!}. \qquad (2.8) \end{aligned}$$

We note that the Wgh function defined by (2.8) is analytic

(i) $\forall z \in \mathbb{C}$ if $1+\sum\limits_{i=1}^{q} B_i - \sum\limits_{i=1}^{p} A_i > 0$

(ii) for $|z| < \dfrac{\prod\limits_{i=1}^{q} (B_i)^{B_i}}{\prod\limits_{i=1}^{p} (A_i)^{A_i}}$ if $1+\sum\limits_{i=1}^{q} B_i - \sum\limits_{i=1}^{p} A_i = 0$

(iii) for $|z| = \dfrac{\prod\limits_{i=1}^{q} (B_i)^{B_i}}{\prod\limits_{i=1}^{p} (A_i)^{A_i}}$ if $1+\sum\limits_{i=1}^{q} B_i - \sum\limits_{i=1}^{p} A_i = 0$ and $\Re\left(\sum\limits_{i=1}^{q} b_i - \sum\limits_{i=1}^{p} a_i\right) + \frac{p-q}{2} > \frac{1}{2}$ (for details one may refer to Kilbas et al. [8]). Also we have

$$\frac{\prod\limits_{i=1}^{q}\Gamma(b_i)}{\prod\limits_{i=1}^{p}\Gamma(a_i)}\, {}_p\Psi_q\left[\begin{array}{c}(a_i,1)_{1,p}\\(b_i,1)_{1,q}\end{array};z\right] = {}_pF_q((a_i)_{1,p};(b_i)_{1,q};z) \tag{2.9}$$

$$= \sum_{n=0}^{\infty}\frac{\prod\limits_{i=1}^{p}(a_i)_n}{\prod\limits_{i=1}^{q}(b_i)_n}\frac{z^n}{n!},\ p \le 1+q,$$

a generalized hypergeometric function which is analytic $\forall z \in \mathbb{C}$ if $p < 1+q$ and $|z| < 1$ if $p = 1+q$ and for $|z| = 1$ if $\Re\left(\sum\limits_{i=1}^{q} b_i - \sum\limits_{i=1}^{p} a_i\right) > 0$. The symbol $(\lambda)_n$ is called Pochhammer symbol defined as:

$$(\lambda)_n = \frac{\Gamma(\lambda+n)}{\Gamma(\lambda)} = \lambda(\lambda+1)...(\lambda+n-1)\,\text{and}\,(\lambda)_0 = 1.$$

While several attempts have been made to study certain mappings of harmonic univalent functions involving various type of calculus operators in the unit disk $\mathcal{U}$ in numerous papers ([1], [2] etc.), not much attention has been paid to study the involvement of any kind of operator for harmonic functions defined in exterior of the unit disk $\widetilde{\mathcal{U}}$. The purpose of this paper is to study certain mappings of a harmonic meromorphic function which involve m-tuple integral operators defined in the exterior of the unit disk $\widetilde{\mathcal{U}}$. For this purpose, an integral operator for a function analytic in $\widetilde{\mathcal{U}}$ is defined as an analogue of the Erdélyi-Kober integral operator [9]. By repeating such m integral operators for h and g of the form (1.2), the m-tuple integral operators are defined. Involving these m-tuple integral operators, an harmonic meromorphic function $\mathbf{F} = \mathbf{H} + \overline{\mathbf{G}} \in \mathcal{MH}$ (defined by (2.5)) is considered and using the results of

Jahangiri [7], necessary and sufficient coefficient conditions for $\mathbf{F}$ to be in the classes $\mathcal{MH}^*(\gamma)$ and $\mathcal{KH}(\gamma)$, respectively, are obtained. Further, with the use of these coefficient conditions, some inequalities (we call them *Wgh inequalities*), involving Wgh functions with their validity conditions, ensuring $\mathbf{F}$ to be in the classes $\mathcal{MH}^*(\gamma)$ and $\mathcal{KH}(\gamma)$ respectively are derived. It is also shown that these Wgh inequalities are necessary for some $\mathbf{F} \in \overline{\mathcal{MH}}$. Bounds and extreme points for functions in $\overline{\mathcal{MH}}^*(\gamma)$ are also obtained.

3. Main Results

In this section, by applying Lemmas 1 and 2, we directly get coefficient conditions for $\mathbf{F} = \mathbf{H} + \overline{\mathbf{G}} \in \mathcal{MH}$ defined by (2.5) to be in the classes $\mathcal{MH}^*(\gamma)$ and $\mathcal{KH}(\gamma)$, respectively as below.

Theorem 1. *If $\mathbf{F} = \mathbf{H} + \overline{\mathbf{G}} \in \mathcal{MH}$ defined by (2.5) satisfies*

$$\sum_{n=1}^{\infty}\Big((n+\gamma)\,|a_n|\,\theta_n + (n-\gamma)\,|b_n|\,\theta_n^{'}\Big) \le (1-\gamma)\,|\alpha|\,\lambda_1 - (1+\gamma)\,|\beta|\,\lambda_1^{'}, \quad (3.1)$$

for $0 \le \gamma < \frac{|\alpha|\lambda_1 - |\beta|\lambda_1^{'}}{|\alpha|\lambda_1 + |\beta|\lambda_1^{'}}$, then $\mathbf{F}$ is harmonic, univalent, orientation-preserving in $\widetilde{\mathcal{U}}$ and $\mathbf{F} \in \mathcal{MH}^*(\gamma)$. The condition (3.1) is necessary if for $f = h + \overline{g} \in \overline{\mathcal{MH}}$, $\mathbf{F} = \mathbf{H} + \overline{\mathbf{G}}$ defined by (2.5) belongs to $\overline{\mathcal{MH}}^*(\gamma)$.

Theorem 2. *If $\mathbf{F} = \mathbf{H} + \overline{\mathbf{G}} \in \mathcal{MH}$ defined by (2.5) satisfies*

$$\sum_{n=1}^{\infty} n\Big((n+\gamma)\,|a_n|\,\theta_n + (n-\gamma)\,|b_n|\,\theta_n^{'}\Big) \le (1-\gamma)\,|\alpha|\,\lambda_1 - (1+\gamma)\,|\beta|\,\lambda_1^{'}, \quad (3.2)$$

for $0 \le \gamma < \frac{|\alpha|\lambda_1 - |\beta|\lambda_1^{'}}{|\alpha|\lambda_1 + |\beta|\lambda_1^{'}}$, then $\mathbf{F} \in \mathcal{KH}(\gamma)$. The condition (3.2) is necessary if for $f = h + \overline{g} \in \overline{\mathcal{MH}}$, $\mathbf{F} = \mathbf{H} + \overline{\mathbf{G}}$ defined by (2.5) belongs to $\overline{\mathcal{KH}}(\gamma)$.

Theorem 3. *Let for $f = h + \overline{g} \in \mathcal{MH}$, h and g are of the form (1.2) satisfying*

$$\sum_{n=1}^{\infty}\left(|a_n| + |b_n|\right) \le |\alpha| - |\beta|\,, \quad (3.3)$$

$\mathbf{F} = \mathbf{H} + \overline{\mathbf{G}} \in \mathcal{MH}$ *be defined by (2.5). If under the same parametric conditions considered in (2.5) together with the validity conditions* $\sum_{i=1}^{m} \delta_i > 2$, $\sum_{i=1}^{m} \delta_i^{'} > 2$, *for* $k = 1, 2$, *the Wgh functions*

$$\Psi_k \ := \ {}_{m+1}\Psi_m \left[\begin{array}{c} (k,1), (\nu_i + 1 + k\beta_i, \beta_i)_{1,m} \\ (\nu_i + 1 + \delta_i + k\beta_i, \beta_i)_{1,m} \end{array} ;1 \right], \tag{3.4}$$

$$\Psi_k' \ := \ {}_{m+1}\Psi_m \left[\begin{array}{c} (k,1), (\nu_i^{'} + 1 + k\beta_i^{'}, \beta_i^{'})_{1,m} \\ (\nu_i^{'} + 1 + \delta_i^{'} + k\beta_i^{'}, \beta_i^{'})_{1,m} \end{array} ;1 \right] \tag{3.5}$$

satisfy for $0 \le \gamma < \frac{|\alpha|\lambda_1 - |\beta|\lambda_1^{'}}{|\alpha|\lambda_1 + |\beta|\lambda_1^{'}}$, *the Wgh inequality:*

$$\Psi_2 + \Psi_2^{'} + (1+\gamma)\,\Psi_1 + (1-\gamma)\,\Psi_1^{'} \le \frac{(1-\gamma)\,|\alpha|\,\lambda_1 - (1+\gamma)\,|\beta|\,\lambda_1^{'}}{(|\alpha| - |\beta|)}, \tag{3.6}$$

then $\mathbf{F}$ *is harmonic, univalent, orientation-preserving in* $\widetilde{\mathcal{U}}$ *and* $\mathbf{F} \in \mathcal{MH}^*(\gamma)$. *The condition (3.6) is necessary if for* $f = h + \overline{g} \in \overline{\mathcal{MH}}$ *of the form (1.4) with* $\alpha > \beta \ge 0$ *and* $a_n = (|\alpha| - |\beta|)$, $b_n = -(|\alpha| - |\beta|)$, $\mathbf{F} = \mathbf{H} + \overline{\mathbf{G}}$ *defined by (2.5) belongs to* $\overline{\mathcal{MH}}^*(\gamma)$.

Proof. To prove the result, by Theorem 1, we need to show

$$S_1 := \sum_{n=1}^{\infty} \left((n+\gamma)\,|a_n|\,\theta_n + (n-\gamma)\,|b_n|\,\theta_n^{'} \right) \le (1-\gamma)\,|\alpha|\,\lambda_1 - (1+\gamma)\,|\beta|\,\lambda_1^{'}.$$

From (3.3), we have $|a_n| \le |\alpha| - |\beta|$ and $|b_n| \le |\alpha| - |\beta|$ for $n \ge 1$. Hence, we get

$$\begin{aligned} S_1 &\le (|\alpha| - |\beta|) \sum_{n=1}^{\infty} \left((n+\gamma)\,\theta_n + (n-\gamma)\,\theta_n^{'} \right) \\ &= (|\alpha| - |\beta|) \left[\sum_{n=2}^{\infty} (n-1) \left(\theta_n + \theta_n^{'} \right) + (1+\gamma) \sum_{n=1}^{\infty} \theta_n + (1-\gamma) \sum_{n=1}^{\infty} \theta_n^{'} \right], \end{aligned}$$

and for $k = 1, 2$, using the identities

$$\sum_{n=k}^{\infty} (n-k+1)_{k-1}\ \theta_n \ = \ \Psi_k, \tag{3.7}$$

$$\sum_{n=k}^{\infty} (n-k+1)_{k-1}\ \theta_n^{'} \ = \ \Psi_k^{'}, \tag{3.8}$$

we obtain

$$\begin{aligned} S_1 &\leq (|\alpha| - |\beta|) \left[\Psi_2 + \Psi_2^{'} + (1+\gamma)\,\Psi_1 + (1-\gamma)\,\Psi_1^{'}\right] \\ &\leq (1-\gamma)\,|\alpha|\,\lambda_1 - (1+\gamma)\,|\beta|\,\lambda_1^{'}, \end{aligned}$$

if (3.6) holds. We note that the validity conditions ensure the convergence of Ψ_k, Ψ_k' for $k = 1, 2$. This proves the sufficient part. Further, by Theorem 1, if $\mathbf{F} \in \overline{\mathcal{MH}}^*(\gamma)$, having $\alpha > \beta \geq 0$ and $a_n = (|\alpha| - |\beta|)$, $b_n = -\,(|\alpha| - |\beta|)$, we get for $0 \leq \gamma < \frac{|\alpha|\lambda_1 - |\beta|\lambda_1^{'}}{|\alpha|\lambda_1 + |\beta|\lambda_1^{'}}$,

$$(|\alpha| - |\beta|) \sum_{n=1}^{\infty} \left((n+\gamma)\,\theta_n + (n-\gamma)\,\theta_n^{'}\right) \leq (1-\gamma)\,|\alpha|\,\lambda_1 - (1+\gamma)\,|\beta|\,\lambda_1^{'},$$

which is equivalent to the Wgh inequality (3.6). This proves Theorem 3. $\square$

Theorem 4. *Let for $f = h + \overline{g} \in \mathcal{MH}$, h and g be of the form (1.2) satisfying (3.3), $\mathbf{F} = \mathbf{H} + \overline{\mathbf{G}} \in \mathcal{MH}$, be defined by (2.5). If under the same parametric conditions considered in (2.5) together with the validity conditions $\sum_{i=1}^{m} \delta_i > 3$, $\sum_{i=1}^{m} \delta_i^{'} > 3$, for $k = 1, 2, 3$, Wgh functions Ψ_k, Ψ_k' defined by (3.4), (3.5) satisfy for $0 \leq \gamma < \frac{|\alpha|\lambda_1 - |\beta|\lambda_1^{'}}{|\alpha|\lambda_1 + |\beta|\lambda_1^{'}}$, the Wgh inequality*

$$\begin{aligned} &\Psi_3 + \Psi_3^{'} + (3+\gamma)\,\Psi_2 + (3-\gamma)\,\Psi_2^{'} + (1+\gamma)\,\Psi_1 + (1-\gamma)\,\Psi_1^{'} \\ \leq\ & \frac{(1-\gamma)\,|\alpha|\,\lambda_1 - (1+\gamma)\,|\beta|\,\lambda_1^{'}}{(|\alpha| - |\beta|)}, \end{aligned} \tag{3.9}$$

then $\mathbf{F} \in \mathcal{KH}(\gamma)$. The condition (3.9) is necessary if for $f = h + \overline{g} \in \overline{\mathcal{MH}}$ of the form (1.4) with $\alpha > \beta \geq 0$ and $a_n = (|\alpha| - |\beta|)$, $b_n = -\,(|\alpha| - |\beta|)$, $\mathbf{F} = \mathbf{H} + \overline{\mathbf{G}}$ defined by (2.5) belongs to $\overline{\mathcal{KH}}(\gamma)$.

Proof. To prove the result, we use Theorem 2, hence, we need to show

$$\begin{aligned} S_2 := \sum_{n=1}^{\infty} n \left((n+\gamma)\,|a_n|\,\theta_n + (n-\gamma)\,|b_n|\,\theta_n^{'}\right) \\ \leq (1-\gamma)\,|\alpha|\,\lambda_1 - (1+\gamma)\,|\beta|\,\lambda_1^{'}. \end{aligned}$$

By (3.3), we have $|a_n| \le |\alpha| - |\beta|$ and $|b_n| \le |\alpha| - |\beta|$ for $n \ge 1$. Hence, using identities (3.7), (3.8) for $k = 1, 2, 3$, we get

$$\begin{aligned}
S_2 &\le (|\alpha| - |\beta|) \sum_{n=1}^{\infty} n \left((n+\gamma)\, \theta_n + (n-\gamma)\, \theta_n^{'} \right) \\
&= (|\alpha| - |\beta|) \left[\sum_{n=3}^{\infty} (n-2)(n-1) \left(\theta_n + \theta_n^{'} \right) \right. \\
&\quad \left. + \sum_{n=2}^{\infty} (n-1) \left((3+\gamma)\, \theta_n + (3-\gamma)\, \theta_n^{'} \right) + \sum_{n=1}^{\infty} \left((1+\gamma)\, \theta_n + (1-\gamma)\, \theta_n^{'} \right) \right] \\
&= (|\alpha| - |\beta|) \left[\Psi_3 + \Psi_3^{'} + (3+\gamma)\, \Psi_2 + (3-\gamma)\, \Psi_2^{'} + (1+\gamma)\, \Psi_1 + (1-\gamma)\, \Psi_1^{'} \right] \\
&\le (1-\gamma)\, |\alpha|\, \lambda_1 - (1+\gamma)\, |\beta|\, \lambda_1^{'},
\end{aligned}$$

if (3.9) holds. This proves that $\mathbf{F} \in \mathcal{KH}(\gamma)$, $0 \le \gamma < \frac{|\alpha|\lambda_1 - |\beta|\lambda_1^{'}}{|\alpha|\lambda_1 + |\beta|\lambda_1^{'}}$. Further, by Theorem 2 if $\mathbf{F}$ defined by (2.5) having $\alpha > \beta \ge 0$ and $a_n = (|\alpha| - |\beta|)$, $b_n = -\,(|\alpha| - |\beta|)$, belongs to $\overline{\mathcal{KH}}(\gamma)$, we get for $0 \le \gamma < \frac{|\alpha|\lambda_1 - |\beta|\lambda_1^{'}}{|\alpha|\lambda_1 + |\beta|\lambda_1^{'}}$,

$$(|\alpha| - |\beta|) \sum_{n=1}^{\infty} n \left((n+\gamma)\, \theta_n + (n-\gamma)\, \theta_n^{'} \right) \le (1-\gamma)\, |\alpha|\, \lambda_1 - (1+\gamma)\, |\beta|\, \lambda_1^{'},$$

which is equivalent to the Wgh inequality (3.9). This proves Theorem 4. $\square$

Taking $\beta_i = 1 = \beta_i^{'}$, $i = 1, 2, ..., m$, we see from (3.4) and (3.5) that for any $k = 1, 2, 3, ..,$

$$\begin{aligned}
\Psi_k &= \Gamma(k) \prod_{i=1}^{m} \frac{\Gamma(\nu_i + k + 1)}{\Gamma(\nu_i + k + \delta_i + 1)}\; F_k, \sum_{i=1}^{m} \delta_i > k, \\
\Psi_k^{'} &= \Gamma(k) \prod_{i=1}^{m} \frac{\Gamma\left(\nu_i^{'} + k + 1\right)}{\Gamma\left(\nu_i^{'} + k + \delta_i^{'} + 1\right)}\; F_k^{'}, \sum_{i=1}^{m} \delta_i^{'} > k,
\end{aligned}$$

where

$$F_k := {}_{m+1}F_m(k, (\nu_i + k + 1)_{1,m}; (\nu_i + \delta_i + k + 1)_{1,m}; 1) \tag{3.10}$$

$$F_k^{'} := {}_{m+1}F_m(k, (\nu_i^{'} + k + 1)_{1,m}; (\nu_i^{'} + \delta_i^{'} + k + 1)_{1,m}; 1). \tag{3.11}$$

Thus, we obtain the following special case of the results proved in Theorems 3 and 4:

Corollary 1. *Let for $f = h + \overline{g} \in \mathcal{MH}$ of the form (1.2) satisfying (3.3) and for some $m \in \mathbb{N}, \delta_i, \delta_i^{'} \in \mathbb{R}_+ \cup \{0\}, \nu_i, \nu_i^{'} \in \mathbb{R}_+, \forall\, i = 1, 2, ..., m$, involving m-tuple integral operators $J_{(1),m}^{(\nu_i),(\delta_i)} \equiv J_m^{(\nu_i),(\delta_i)}$ and $J_{(1),m}^{(\nu_i^{'}),(\delta_i^{'})} \equiv J_m^{(\nu_i^{'}),(\delta_i^{'})}$ as defined by (2.4),*

$$\mathbf{F}_1 = \mathbf{H}_1 + \overline{\mathbf{G}_1} \in \mathcal{MH}, \tag{3.12}$$

where

$$\begin{aligned}
\mathbf{H}_1(z) &= J_m^{(\nu_i),(\delta_i)} h(z) = \alpha\mu_1 z + \sum_{n=1}^{\infty} \sigma_n\ a_n\ z^{-n}, \\
\mathbf{G}_1(z) &= J_m^{(\nu_i^{'}),(\delta_i^{'})} g(z) = \beta\mu_1^{'} z + \sum_{n=1}^{\infty} \sigma_n^{'}\ b_n\ z^{-n}, \mu_1^{'} < \mu_1,
\end{aligned}$$

$$\begin{aligned}
\sigma_n &= \prod_{i=1}^{m} \frac{\Gamma(\nu_i + 1 + n)}{\Gamma(\nu_i + \delta_i + 1 + n)}, \sigma_n^{'} = \prod_{i=1}^{m} \frac{\Gamma\left(\nu_i^{'} + 1 + n\right)}{\Gamma\left(\nu_i^{'} + \delta_i^{'} + 1 + n\right)}, n \geq 1, \\
\mu_1 &= \prod_{i=1}^{m} \frac{\Gamma(\nu_i)}{\Gamma(\nu_i + \delta_i)}, \mu_1^{'} = \prod_{i=1}^{m} \frac{\Gamma\left(\nu_i^{'}\right)}{\Gamma\left(\nu_i^{'} + \delta_i^{'}\right)}.
\end{aligned}$$

together with the validity conditions $\sum_{i=1}^{m} \delta_i > 2$, $\sum_{i=1}^{m} \delta_i^{'} > 2$, for $k = 1, 2$, functions $F_k, F_k^{'}$ defined by (3.10), (3.11) satisfy for $0 \leq \gamma < \frac{|\alpha|\mu_1 - |\beta|\mu_1^{'}}{|\alpha|\mu_1 + |\beta|\mu_1^{'}}$, the inequality

$$\begin{aligned}
&\prod_{i=1}^{m} \frac{\Gamma(\nu_i + 2)}{\Gamma(\nu_i + \delta_i + 2)} \left(\prod_{i=1}^{m} \frac{(\nu_i + 2)}{(\nu_i + \delta_i + 2)} F_2 + (1+\gamma)\ F_1 \right) + \\
&\prod_{i=1}^{m} \frac{\Gamma\left(\nu_i^{'} + 2\right)}{\Gamma\left(\nu_i^{'} + \delta_i^{'} + 2\right)} \left(\prod_{i=1}^{m} \frac{\left(\nu_i^{'} + 2\right)}{\left(\nu_i^{'} + \delta_i^{'} + 2\right)}\ F_2^{'} + (1-\gamma\ )\ F_1^{'} \right) \\
\leq\ & \frac{(1-\gamma)\,|\alpha|\,\mu_1 - (1+\gamma)\,|\beta|\,\mu_1^{'}}{(|\alpha| - |\beta|)},
\end{aligned} \tag{3.13}$$

then $\mathbf{F}_1$ is harmonic, univalent, orientation-preserving in $\widetilde{\mathcal{U}}$ and $\mathbf{F}_1 \in \mathcal{MH}^*(\gamma)$. The condition (3.13) is necessary if for $f = h + \overline{g} \in \overline{\mathcal{MH}}$ of the form (1.4) with $\alpha > \beta \geq 0$ and $a_n = (|\alpha| - |\beta|)$, $b_n = -(|\alpha| - |\beta|)$, $\mathbf{F}_1$ defined by (3.12) belongs to $\overline{\mathcal{MH}}^*(\gamma)$.

Corollary 2. *Let for $f = h + \overline{g} \in \mathcal{MH}$ of the form (1.2) satisfying (3.3), $\mathbf{F}_1 = \mathbf{H}_1 + \overline{\mathbf{G}_1} \in \mathcal{MH}$ defined by (3.12). If for $\sum_{i=1}^{m} \delta_i > 3$, $\sum_{i=1}^{m} \delta_i^{'} > 3$ and for $k = 1, 2, 3$, functions $F_k, F_k^{'}$ defined by (3.10), (3.11) satisfy for $0 \leq \gamma < \frac{|\alpha|\mu_1 - |\beta|\mu_1^{'}}{|\alpha|\mu_1 + |\beta|\mu_1^{'}}$, the inequality*

$$\begin{aligned}
&\prod_{i=1}^{m} \frac{\Gamma(\nu_i + 2)}{\Gamma(\nu_i + \delta_i + 2)} \left(2 \prod_{i=1}^{m} \frac{(\nu_i + 3)(\nu_i + 2)}{(\nu_i + \delta_i + 3)(\nu_i + \delta_i + 2)} F_3 + \right. \\
&\left. (3 + \gamma) \prod_{i=1}^{m} \frac{(\nu_i + 2)}{(\nu_i + \delta_i + 2)} F_2 + (1 + \gamma) F_1 \right) + \\
&\prod_{i=1}^{m} \frac{\Gamma\left(\nu_i^{'} + 2\right)}{\Gamma\left(\nu_i^{'} + \delta_i^{'} + 2\right)} \left(2 \prod_{i=1}^{m} \frac{\left(\nu_i^{'} + 3\right)\left(\nu_i^{'} + 2\right)}{\left(\nu_i^{'} + \delta_i^{'} + 3\right)\left(\nu_i^{'} + \delta_i^{'} + 2\right)} F_3^{'} + \right. \\
&\left. (3 - \gamma) \prod_{i=1}^{m} \frac{\left(\nu_i^{'} + 2\right)}{\left(\nu_i^{'} + \delta_i^{'} + 2\right)} F_2^{'} + (1 - \gamma) F_1^{'} \right) \\
\leq \; & \frac{(1 - \gamma)|\alpha|\mu_1 - (1 + \gamma)|\beta|\mu_1^{'}}{(|\alpha| - |\beta|)},
\end{aligned} \tag{3.14}$$

then $\mathbf{F}_1 \in \mathcal{KH}(\gamma)$. The condition (3.14) is necessary if $\mathbf{F}_1$ defined by (3.12) with $a_n = (|\alpha| - |\beta|)$ and $b_n = -(|\alpha| - |\beta|)$, belongs to $\overline{\mathcal{KH}}(\gamma)$.

4. Bounds and Extreme Points

As $\overline{\mathcal{KH}}(\gamma) \subset \overline{\mathcal{MH}}^*(\gamma)$, we obtain bounds for the functions $\mathbf{F} = \mathbf{H} + \overline{\mathbf{G}} \in \overline{\mathcal{MH}}^*(\gamma)$, $0 \leq \gamma < \frac{|\alpha|\lambda_1 - |\beta|\lambda_1^{'}}{|\alpha|\lambda_1 + |\beta|\lambda_1^{'}}$, and extreme points for $\overline{\mathcal{MH}}^*(\gamma)$.

Theorem 5. *Let for $\mathbf{F} = \mathbf{H} + \overline{\mathbf{G}} \in \overline{\mathcal{MH}}^*(\gamma)$, $0 \leq \gamma < \frac{|\alpha|\lambda_1 - |\beta|\lambda_1^{'}}{|\alpha|\lambda_1 + |\beta|\lambda_1^{'}}$, then for $|z| > r > 1$,*

$$|\mathbf{F}(z)| \leq \left(|\alpha|\lambda_1 + |\beta|\lambda_1^{'}\right) r + \frac{r^{-1}\left((1 - \gamma)|\alpha|\lambda_1 - (1 + \gamma)|\beta|\lambda_1^{'}\right)}{(1 - \gamma)},$$

$$|\mathbf{F}(z)| \geq \left(|\alpha|\lambda_1 - |\beta|\lambda_1^{'}\right) r - \frac{r^{-1}\left((1 - \gamma)|\alpha|\lambda_1 - (1 + \gamma)|\beta|\lambda_1^{'}\right)}{(1 - \gamma)}.$$

Proof. Let for $f = h + \overline{g} \in \overline{\mathcal{MH}}$ of the form (1.4), $\mathbf{F} = \mathbf{H} + \overline{\mathbf{G}} \in \overline{\mathcal{MH}}$, be of the form (2.5) belongs to $\overline{\mathcal{MH}}^*(\gamma)$. Applying Theorem 1 and taking the absolute value of $\mathbf{F}$, we get

$$\begin{aligned} |\mathbf{F}(z)| &= \left| |\alpha| \lambda_1 z + |\beta| \lambda_1^{'} \overline{z} + \sum_{n=1}^{\infty} \theta_n |a_n| z^{-n} - \overline{\sum_{n=1}^{\infty} \theta_n^{'} |b_n| z^{-n}} \right| \\ &\leq \left(|\alpha| \lambda_1 + |\beta| \lambda_1^{'} \right) r + \sum_{n=1}^{\infty} \left(\theta_n |a_n| + \theta_n^{'} |b_n| \right) r^{-n} \\ &\leq \left(|\alpha| \lambda_1 + |\beta| \lambda_1^{'} \right) r + \frac{r^{-1}}{(1-\gamma)} \sum_{n=1}^{\infty} \left((n+\gamma) \theta_n |a_n| + (n-\gamma) \theta_n^{'} |b_n| \right) \\ &\leq \left(|\alpha| \lambda_1 + |\beta| \lambda_1^{'} \right) r + \frac{r^{-1} \left((1-\gamma) |\alpha| \lambda_1 - (1+\gamma) |\beta| \lambda_1^{'} \right)}{(1-\gamma)}. \end{aligned}$$

Similarly, lower bound can be obtained. This proves Theorem 5. □

On taking $\delta_i = 0$, for each $i = 1, 2, ..., m$, in Theorem 5, we get, following corollary:

Corollary 3. *If $f = h + \overline{g} \in \overline{\mathcal{MH}}^*(\gamma)$, $0 \leq \gamma < \frac{|\alpha| - |\beta|}{|\alpha| + |\beta|}$, then for $z \in \widetilde{\mathcal{U}}$,*

$$\begin{aligned} (|\alpha| - |\beta|) r - \frac{r^{-1} ((1-\gamma) |\alpha| - (1+\gamma) |\beta|)}{(1-\gamma)} &\leq |f(z)| \\ &\leq (|\alpha| + |\beta|) r + \frac{r^{-1} ((1-\gamma) |\alpha| - (1+\gamma) |\beta|)}{(1-\gamma)}. \end{aligned}$$

Remark 1. *A result on bounds obtained in [6] follows on putting $\gamma = 0$ in the above corollary.*

Theorem 6. *For $f = h + \overline{g} \in \overline{\mathcal{MH}}$ of the form (1.4), $\mathbf{F} = \mathbf{H} + \overline{\mathbf{G}}$ defined by (2.5) belongs to $\overline{\mathcal{MH}}^*(\gamma)$, $0 \leq \gamma < \frac{|\alpha|\lambda_1 - |\beta|\lambda_1^{'}}{|\alpha|\lambda_1 + |\beta|\lambda_1^{'}}$, if and only if f can be expressed for $z \in \widetilde{\mathcal{U}}$, as*

$$f(z) = \sum_{n=0}^{\infty} (x_n h_n + y_n g_n),$$

where

$$h_0=|\alpha|\,z+|\beta|\,\overline{z},\ h_n=|\alpha|\,z+|\beta|\,\overline{z}+\frac{\left((1-\gamma)\,|\alpha|\,\lambda_1-(1+\gamma)\,|\beta|\,\lambda_1^{'}\right)z^{-n}}{(n+\gamma)\,\theta_n},n\geq 1, \tag{4.1}$$

$$g_0=|\alpha|\,z+|\beta|\,\overline{z},\ g_n=|\alpha|\,z+|\beta|\,\overline{z}-\frac{\left((1-\gamma)\,|\alpha|\,\lambda_1-(1+\gamma)\,|\beta|\,\lambda_1^{'}\right)\overline{z}^{-n}}{(n-\gamma)\,\theta_n^{'}},n\geq 1, \tag{4.2}$$

and

$$\sum_{n=0}^{\infty}(x_n+y_n)=1,\quad x_n,\,y_n\geq 0\ .$$

The extreme points of $\overline{\mathcal{MH}}^{*}(\gamma)$ are h_n and g_n given by (4.1) and (4.2).

Proof. Let

$$\begin{aligned}
f(z) &= \sum_{n=0}^{\infty}(x_nh_n+y_ng_n)\\
&= x_0h_0+y_0g_0\\
&\quad+\sum_{n=1}^{\infty}\left(x_n\left(|\alpha|\,z+|\beta|\,\overline{z}+\frac{\left((1-\gamma)\,|\alpha|\,\lambda_1-(1+\gamma)\,|\beta|\,\lambda_1^{'}\right)z^{-n}}{(n+\gamma)\,\theta_n}\right)\right.\\
&\quad\left.+y_n\left(|\alpha|\,z+|\beta|\,\overline{z}-\frac{\left((1-\gamma)\,|\alpha|\,\lambda_1-(1+\gamma)\,|\beta|\,\lambda_1^{'}\right)\overline{z}^{-n}}{(n-\gamma)\,\theta_n^{'}}\right)\right)\\
&= |\alpha|\,z+|\beta|\,\overline{z}+\sum_{n=1}^{\infty}\frac{\left((1-\gamma)\,|\alpha|\,\lambda_1-(1+\gamma)\,|\beta|\,\lambda_1^{'}\right)x_n}{(n+\gamma)\,\theta_n}z^{-n}\\
&\quad-\sum_{n=1}^{\infty}\frac{\left((1-\gamma)\,|\alpha|\,\lambda_1-(1+\gamma)\,|\beta|\,\lambda_1^{'}\right)y_n}{(n-\gamma)\,\theta_n^{'}}\overline{z}^{-n}.
\end{aligned}$$

By Theorem 1, it is observed that

$$\begin{aligned}
& \sum_{n=1}^{\infty}\frac{(n+\gamma)\,\theta_n}{\left((1-\gamma)\,|\alpha|\,\lambda_1-(1+\gamma)\,|\beta|\,\lambda_1'\right)}\left|\frac{\left((1-\gamma)\,|\alpha|\,\lambda_1-(1+\gamma)\,|\beta|\,\lambda_1'\right)x_n}{(n+\gamma)\,\theta_n}\right| \\
& +\sum_{n=1}^{\infty}\frac{(n-\gamma)\,\theta_n'}{\left((1-\gamma)\,|\alpha|\,\lambda_1-(1+\gamma)\,|\beta|\,\lambda_1'\right)}\left|\frac{\left((1-\gamma)\,|\alpha|\,\lambda_1-(1+\gamma)\,|\beta|\,\lambda_1'\right)y_n}{(n-\gamma)\,\theta_n'}\right| \\
= & \sum_{n=1}^{\infty}(x_n+y_n)=1-(x_0+y_0)\le 1,
\end{aligned}$$

hence, $\mathbf{F}\in\overline{\mathcal{MH}}^*(\gamma)$. Conversely, suppose that for $f=h+\overline{g}\in\overline{\mathcal{MH}}$ of the form (1.4), $\mathbf{F}=\mathbf{H}+\overline{\mathbf{G}}$ of the form (2.5) belongs to $\overline{\mathcal{MH}}^*(\gamma)$, $0\le\gamma<\frac{|\alpha|\lambda_1-|\beta|\lambda_1'}{|\alpha|\lambda_1+|\beta|\lambda_1'}$, then inequality (3.1) holds. Setting

$$\begin{aligned}
x_n &= \frac{(n+\gamma)\,\theta_n}{\left((1-\gamma)\,|\alpha|\,\lambda_1-(1+\gamma)\,|\beta|\,\lambda_1'\right)}\,|a_n|\,, \\
y_n &= \frac{(n-\gamma)\,\theta_n'}{\left((1-\gamma)\,|\alpha|\,\lambda_1-(1+\gamma)\,|\beta|\,\lambda_1'\right)}\,|b_n|\,,
\end{aligned}$$

we get

$$\begin{aligned}
f(z) &= |\alpha|\,z+|\beta|\,\overline{z}+\sum_{n=1}^{\infty}|a_n|\,z^{-n}-\sum_{n=1}^{\infty}|b_n|\,\overline{z}^{-n} \\
&= |\alpha|\,z+|\beta|\,\overline{z}+\sum_{n=1}^{\infty}\left(\frac{\left((1-\gamma)\,|\alpha|\,\lambda_1-(1+\gamma)\,|\beta|\,\lambda_1'\right)x_n}{(n+\gamma)\,\theta_n}\right)z^{-n} \\
&\quad -\sum_{n=1}^{\infty}\left(\frac{\left((1-\gamma)\,|\alpha|\,\lambda_1-(1+\gamma)\,|\beta|\,\lambda_1'\right)y_n}{(n-\gamma)\,\theta_n'}\right)\overline{z}^{-n} \\
&= |\alpha|\,z+|\beta|\,\overline{z}+\sum_{n=1}^{\infty}\left[h_n-(|\alpha|\,z+|\beta|\,\overline{z})\right]x_n+\sum_{n=1}^{\infty}\left[g_n-(|\alpha|\,z+|\beta|\,\overline{z})\right]y_n \\
&= (|\alpha|\,z+|\beta|\,\overline{z})\left[1-\sum_{n=1}^{\infty}(x_n+y_n)\right]+\sum_{n=1}^{\infty}x_nh_n+\sum_{n=1}^{\infty}y_ng_n \\
&= \sum_{n=0}^{\infty}(x_nh_n+y_ng_n)\,.
\end{aligned}$$

This proves Theorem 6. $\square$

On taking $\delta_i = 0$, for each $i = 1, 2, ..., m$, in Theorem 6, we get, following corollary:

Corollary 4. *A harmonic function $f = h + \overline{g}$ of the form (1.4) belongs to $\overline{\mathcal{MH}}^*(\gamma)$, $0 \leq \gamma < \frac{|\alpha|-|\beta|}{|\alpha|+|\beta|}$, if and only if f can be expressed for $z \in \widetilde{\mathcal{U}}$, as*

$$f(z) = \sum_{n=0}^{\infty} (x_n h_n + y_n g_n),$$

where

$$h_0 = |\alpha|\, z + |\beta|\, \overline{z},\ h_n = |\alpha|\, z + |\beta|\, \overline{z} + \frac{((1-\gamma)\,|\alpha| - (1+\gamma)\,|\beta|)\, z^{-n}}{(n+\gamma)}, n \geq 1, \quad (4.3)$$

$$g_0 = |\alpha|\, z + |\beta|\, \overline{z},\ g_n = |\alpha|\, z + |\beta|\, \overline{z} - \frac{((1-\gamma)\,|\alpha| - (1+\gamma)\,|\beta|)\, \overline{z}^{-n}}{(n-\gamma)}, n \geq 1, \quad (4.4)$$

and

$$\sum_{n=0}^{\infty} (x_n + y_n) = 1, \quad x_n, y_n \geq 0 \ .$$

The extreme points of $\overline{\mathcal{MH}}^*(\gamma)$ in this case are h_n and g_n given by (4.3) and (4.4).

References

[1] O.P. Ahuja, Planer harmonic convolution operators generated by hypergeometric functions, *Integ. Trans. Spec. Funct.*, **18**, No. 3 (2007), 165-177.

[2] O.P. Ahuja, Harmonic starlike and convexity of integral operators generated by hypergeometric series, *Integ. Trans. Spec. Funct.*, **20**, No. 8 (2009), 629-641.

[3] J. Clunie, T. Sheil-Ssmll, Harmonic univalent functions, *Ann. Acad. Sci. Fenn. Ser. A. I. Math.*, **9** (1984), 3-25.

[4] A. Erdélyi, *Higher Trancendental Functions*, Bateman Manuscript Project, Volume **1**, McGraw Hill (1958).

[5] W. Hengartner, G. Schober, Univalent harmonic functions, *Trans. Amer. Math. Soc.*, **299**, No. 1 (1987), 1-31.

[6] J.M. Jahangiri, H. Silverman, Meromorphic univalent harmonic functions with negative coefficients, *Bull. Korean Math. Soc.*, **36**, No. 4 (1999), 763-770.

[7] J.M. Jahangiri, Harmonic meromorphic starlike functions, *Bull. Korean Math. Soc.*, **37**, No. 2 (2000), 291-301.

[8] A.A. Kilbas, M. Saigo, J.J. Trujillo, On the generalized Wright function, *Fract. Calc. Appl. Anal.*, **5**, No. 4 (2002), 437-460.

[9] V.S. Kiryakova, *Generalized Fractional Calculas and Applications*, Pitman Research Notes in Math. Series, **301**, Longman and J. Wiley, Harlow, New York (1994).

[10] V.S. Kiryakova, Fractional integration operators involving Fox's $H^{m,\sigma}_{m,m}$-function, *Comptes rendes de l'Académie Bulgare des Sciences*, **41**, No. 11 (1988), 11-14.

[11] V.S. Kiryakova, Criteria for univalence, convexity and starlikeness, and distortion theorems for generalized fractional integrals involving Fox's H-functions, In: *Proceedings of the First International Conference on Mathematics and Statistics*, Sharjah, U.A.E., March 18-21, 2010.

[12] V.S. Kiryakova, The operators of generalized fractional calculus and their action in classes of univalent functions, In: *Proceedings of International Symposium on Geometric Function Theory and Applications 2010*, Sofia, August 27-31, 2010.

[13] H.M. Srivastava, H.L. Manocha, *A Treatise on Generating Functions*, Halsted Press, Ellis Horwood Limlted and John Wiley and Sons, New York, Chichester, Toronto (1984).